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1990 J. Phys. A: Math. Gen. 23 1127

(http://iopscience.iop.org/0305-4470/23/7/018)

J. Phys. A: Math. Gen. **23** (1990) 1127–1136. Printed in the UK

The fully supersymmetric AKNS equations

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Received 8 May 1989, in final form 30 November 1989

Abstract. The general fully supersymmetric approach to the AKNS method is presented. The supersymmetric KdV and MKdV equations are considered as examples. The supersymmetric Bäcklund transformation is constructed for the supersymmetric KdV equation.

1. Introduction

Almost 10 years ago physicists started investigations of the supersymmetric soliton equations [1-3]. From the mathematical point of view supersymmetry allowed us to extend the class of nonlinear partial differential equations solvable by new methods such as the Lax pair, Bäcklund transformation, etc [4-6]. Mathematically it amounts to incorporating the anticommuting variable of Grassman type together with the usual commuting (c-number) variables. The case of extended supersymmetry seems very promising for soliton theory. For example, $N = 2$ extended supersymmetry is characterised by incorporating two boson and two fermion fields and treats those fields equally. After performing the supersymmetrisation and removing the fermionic fields we obtain the system of interacting bosonic fields or, in mathematical language, the system of interacting partial nonlinear differential equations. We expect that this system shares similar properties to the non-supersymmetric system.

The most important property of the soliton from the field theory point of view is its particle-like behaviour. It is natural to ask: is this property preserved in relativistic supersymmetric field theory [7, 8]? In order to give the affirmative answer one should prepare the supersymmetrisation of the given classical soliton theory. However, we have no unique prescription of how to do this. What we have is formal and incomplete and can be divided into several frameworks such as the geometrical, algebraic or fermionic and fully supersymmetric approach.

In the geometrical framework the soliton equations are considered in the form of the Cartan–Maurer equations on the matrix 1-forms belonging to some Lie algebra of a Lie group. Then the supersymmetrisation is performed by generalisation of the Cartan–Maurer equation to the graded supersymmetric Lie algebra. This method has been used in the Ablowitz–Kaup–Newell–Segur (AKNS) [9] representation in which the SL(2, C) group has been changed to OSP(1, 2) [10] and later to OSP(N, 2) groups [11]. This is the consistent mathematical theory but it is not satisfactory from the physical point of view. For this generalisation the supersymmetry invariance is broken as, for example, in the Korteweg–de Vries equation (KdV). It is due to the method where we add a fermion field 'by hand' in the proper mathematical way.

0305-4470/90/071127+10$03.50

A natural way to extend the given equation to a fully supersymmetric system is simply to rewrite this equation in terms of the superfields and the covariant derivatives. However such a superfield generalisation is not always unique. For example, for the KdV equation this generalisation contains one free parameter [12]. This freedom can be restricted assuming an additional condition or conditions on the system. Now it is well known that the second Hamiltonian structure of the KdV equation is connected with the Virasoro algebra realised in terms of the Poisson brackets. From the knowledge of the supersymmetric extension of the Virasoro algebra and from the supersymmetric extension of its second Hamiltonian in the superfield form it is possible to supersymmetrise the KdV [12-17]. This manner fixed the mentioned free parameter also. The equation obtained is different from the fermionically extended KdV.

The KdV equation generalised in the superfield form possesess the Lax pair but how it is connected with the AKNS representation is the subject of the present paper. Here we present two different manners of the fully supersymmetrisation of the AKNS approach. In the first we generalise the Cartan-Maurer equation by the supersymmetrisation, e.g. use the superfield in the connection form and by addition of one more connection with the opposite parity 'statistic' to the original connection. The assumption of the flatness of our combination of the superconnection gives us the analogue of the fully supersymmetric AKNS equations. In the second way we started our considerations from the fully supersymmetric Riccati equations. It is well known that the AKNS system can be written as the system of two Riccati equations called the projection representation of AKNS [18]. The transition from Riccati equations to the matrix representation is straightforward and here we supersymmetrise this transition. As the result we endow the fully supersymmetric KdV and mKdV to this representation.

The advantage of this method is the possibility to construct the Bäcklund transformation for the equations under consideration. We use the projective representation of the AKNS in this aim and we show that this way is equivalent to the gauge transformation of our supersymmetric connection form.

2. The generalisation of the Cartan–Maurer equation

The setup of the inverse scattering method of AKNS is given by the following completely integrable Pfaffian system

$$-\mathrm{d}V = \Omega V \tag{1}$$

where Ω is a traceless 2×2 matrix depending on the spectral parameter η and on the u function of the x and t variables and on the derivatives of u. The integrability of (1) requires the vanishing of the 2-form

$$\mathrm{dd}V = 0 = \mathrm{d}\Omega\, V - \Omega \wedge \Omega \cdot V. \tag{2}$$

Usually the supersymmetrisation is achieved by replacing a connection 1-form with the 1-form which has the value in the graded supersymmetric group. The fermionic part of the supersymmetry in this approach is added independently from the bosonic part and hence the supersymmetry invariance is broken. We can rescue this invariance by the supersymmetrisation of the function u by the use of a superfield in the Ω form and by addition of one more connection form with the opposite parity to Ω. Indeed let us consider the following system of differential supersymmetric equations:

$$-\mathrm{d}V = \Omega V + \tilde{\Omega}\, \mathrm{D}V \tag{3}$$

where Ω, $\tilde{\Omega}$ are 1-forms depending on the spectral parameter η on the superfunction q of the x and t variables and on the superderivatives of q, V is a super Jost function and the superderivative is defined by

$$\mathrm{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}. \tag{4}$$

The integrability of (3) requires the vanishing of the 2-form

$$\begin{aligned} \mathrm{dd}V = 0 = \mathrm{d}\Omega V + \mathrm{d}\tilde{\Omega}\,\mathrm{D}V - \Omega\wedge\Omega V - \Omega\wedge\tilde{\Omega}\,\mathrm{D}V \\ -\tilde{\Omega}\wedge\mathrm{D}\Omega V - \tilde{\Omega}\wedge\bar{\Omega}\mathrm{D}V - \tilde{\Omega}\wedge\mathrm{D}\tilde{\Omega}\mathrm{D}V \\ -\tilde{\Omega}\wedge\bar{\tilde{\Omega}}V_x \end{aligned} \tag{5}$$

where

$$\bar{\Omega}_{ij} = \delta(\Omega_{ij})\Omega_{ij} \qquad \bar{\tilde{\Omega}}_{ij} = \delta(\tilde{\Omega}_{ij})\tilde{\Omega}_{ij} \tag{6}$$

$$\delta(z) = \begin{cases} -1 & \text{if } z \text{ is odd} \\ 1 & \text{if } z \text{ is even.} \end{cases} \tag{7}$$

Introducing

$$\Omega = \Omega_0\,\mathrm{d}x + \Omega_1\,\mathrm{d}t \tag{8}$$

$$\tilde{\Omega} = \tilde{\Omega}_0\,\mathrm{d}x + \tilde{\Omega}_1\,\mathrm{d}t \tag{9}$$

(5) can be cast as

$$\mathrm{d}\Omega = \Omega\wedge\Omega + \tilde{\Omega}\wedge\mathrm{D}\Omega - \tilde{\Omega}\wedge\bar{\tilde{\Omega}}\cdot\Omega_0 \tag{10}$$

$$\mathrm{d}\tilde{\Omega} = \Omega\wedge\tilde{\Omega} + \tilde{\Omega}\wedge\bar{\Omega} + \tilde{\Omega}\wedge\mathrm{D}\tilde{\Omega} - \tilde{\Omega}\wedge\bar{\tilde{\Omega}}\cdot\tilde{\Omega}_0. \tag{11}$$

Equations (10) and (11) constitute the analogue of the fully supersymmetric AKNS representation. By the proper parametrisation of Ω and $\tilde{\Omega}$ we can obtain the fully supersymmetric equations.

(i) The fully supersymmetric KdV equation, first obtained by Manin and Radul [19], is

$$q_t = q_{xxx} + 3(q\,\mathrm{D}q)_x \tag{12}$$

$$q = q^0 + \theta q' \tag{13}$$

where q^0 is the odd function while q' is the even function of x and t. We then have

$$-\Omega = \begin{pmatrix} \eta & q \\ 0 & -\eta \end{pmatrix}\mathrm{d}x + \begin{pmatrix} 0 & B \\ 2\eta q & -A \end{pmatrix}\mathrm{d}t \tag{14}$$

$$-\tilde{\Omega} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\mathrm{d}x + \begin{pmatrix} A_2 & 0 \\ C & A_2 \end{pmatrix}\mathrm{d}t \tag{15}$$

where η is an arbitrary constant parameter, and

$$B = q_{xx} + 2\eta q_x + 4\eta^2 q + 2q\,\mathrm{D}q \tag{16}$$

$$A_1 = 8\eta^3 + \mathrm{D}q_x + 2\eta\,\mathrm{D}q \tag{17}$$

$$A_2 = 2\eta q + q_x \tag{18}$$

$$C = -4\eta^2 - 2\,\mathrm{D}q. \tag{19}$$

(ii) The fully supersymmetric modified Korteveg–de Vries equation introduced by Sasaki and Yamanaki [14] is

$$q_t = -q_{xxx} + 3q_x \, \mathrm{D}q \, \mathrm{D}q + 3q \, \mathrm{D}q_x \, \mathrm{D}q \tag{20}$$

$$-\Omega = \begin{pmatrix} \eta & q \\ g_1 & -\eta \end{pmatrix} \mathrm{d}x + \begin{pmatrix} h & B \\ g_2 & -A_1 + h \end{pmatrix} \mathrm{d}t \tag{21}$$

$$-\tilde{\Omega} = \begin{pmatrix} 0 & 0 \\ \mathrm{D}q & 0 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} A_2 & 0 \\ C & A_2 \end{pmatrix} \mathrm{d}t \tag{22}$$

where

$$g_1 = -\eta q \qquad\qquad h = 2\eta q q_x \tag{23}$$

$$g_2 = \eta(q_{xx} - 2\eta q_x + 4\eta^2 q - 2 \, \mathrm{D}q \, \mathrm{D}qq) \tag{24}$$

$$B = -q_{xx} - 2\eta q_x - 4\eta^2 q + 2q \, \mathrm{D}q \, \mathrm{D}q \tag{25}$$

$$C = -\mathrm{D}q_{xx} + 2\eta \, \mathrm{D}q_x - 4\eta^2 \, \mathrm{D}q + 2 \, \mathrm{D}q \, \mathrm{D}q \, \mathrm{D}q - 2qq_x \, \mathrm{D}q \tag{26}$$

$$A_1 = 2\eta \, \mathrm{D}q \, \mathrm{D}q - 8\eta^3 \tag{27}$$

$$A_2 = \mathrm{D}(qq_x) + 2\eta \, \mathrm{D}qq. \tag{28}$$

Notice that the 1-forms Ω and $\tilde{\Omega}$ are graded matrices and hence $V = (v_1, v_2)^{\mathrm{T}}$ where v_1 is the odd superfunction while v_2 is even. These 1-forms can always be built in 1-form by the increasing the dimensionality of the connection and by going to the component form of (3). To see it let us use the KdV equation as an example and let us define the 0-form by

$$Z = (v_1', v_2', v_1^0, v_2^0) \tag{29}$$

$$v_1 = v_1^0 + \theta v_1' \qquad\qquad v_2 = v_2' + \theta v_2^0 \tag{30}$$

and 1-form $\tilde{\tilde{\Omega}}$ by

$$-\tilde{\tilde{\Omega}} = \begin{pmatrix} \eta & q^1 & 0 & -q^0 \\ -1 & -\eta & 0 & 0 \\ 0 & q^0 & \eta & 0 \\ 0 & -q^0 & -\eta & -\eta \end{pmatrix} \mathrm{d}x + \begin{pmatrix} A_2^1 & B^1 - A_2^0 q^0 & -\eta A_2^0 & B^0 \\ c^1 & -A_1^1 & 2\eta q^0 & A_2^0 \\ A_2^0 & B^0 & 0 & 0 \\ -q_x^0 & -A_1^0 + \eta A_2^0 + C^1 q^0 & -4\eta^3 & -8\eta^3 \end{pmatrix} \mathrm{d}t. \tag{31}$$

3. The supersymmetric Riccati equation and the supersymmetric projective representation of the AKNS

The Riccati equations which appear in the AKNS representation are in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}x} \Gamma = \Gamma_x = 2\eta \Gamma + R + K\Gamma^2 \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma = \Gamma_t = B + 2A\Gamma - C\Gamma^2 \tag{33}$$

where η is an arbitrary parameter, R and K are functions of some function q of the x and t variable, while A, B, C are the functions of q and their derivatives and also the η parameter. The integrability of (32) and (33) gives us the same equations as in the AKNS approach. For this reason such a representation is called the projective AKNS representation. The transition from the projective representation to its matrix form is achieved if we factorise the function Γ as

$$\Gamma = \psi_1/\psi_2. \tag{34}$$

Introducing (34) to (32) and (33) we obtain

$$\psi_{1x}\psi_2 - \psi_1\psi_{2x} = 2\eta\psi_1\psi_2 + R\psi_2\psi_2 + K\psi_1\psi_1 \tag{35}$$

$$\psi_{1t}\psi_2 - \psi_1\psi_{2t} = B\psi_2\psi_2 + 2A\psi_1\psi_2 - C\psi_1\psi_2. \tag{36}$$

Now this system can be split (not uniquely) in such a way to obtain the AKNS representation

$$\psi_x = \begin{pmatrix} \eta & R \\ K & -\eta \end{pmatrix}\psi = S\psi \tag{37}$$

$$\psi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\psi = W\psi. \tag{38}$$

The integrability of (37) and (38) gives us the partial nonlinear differential equation of the function q. For example, the KdV equation is obtained assuming that

$$R = q \qquad K = 1 \tag{39}$$

$$B = q_{xx} + 2\eta q_x + 4\eta^2 q + 2q^2 \tag{40}$$

$$A = q_x + 2\eta q + 4\eta^3 \tag{41}$$

$$C = -4\eta^2 - 2q \tag{42}$$

$$q_t = q_{xxx} + 6qq_x. \tag{43}$$

Now we are preparing to supersymmetrise this approach. First let us notice that it is possible to write down the fully supersymmetric version of (32) and (33) in terms of a superfield as

$$\Gamma_x = 2\eta\Gamma + R + K\Gamma\, \mathrm{D}\Gamma + N\, \mathrm{D}\Gamma\, \mathrm{D}\Gamma \tag{44}$$

$$\Gamma_t = B + A_1\Gamma + A_2\, \mathrm{D}\Gamma - C\Gamma\, \mathrm{D}\Gamma + M\, \mathrm{D}\Gamma\, \mathrm{D}\Gamma \tag{45}$$

where Γ, R, N, B, A_2, M are odd functions while K, A, A_1, C are even functions. Notice that our supersymmetric Riccati equations are evidently invariant under the supersymmetric transformation

$$\delta = \xi(\partial_\theta - \theta\partial_x) \tag{46}$$

where ξ is a 'small' Grasmann number.

The integrability of (44) and (45) gives us, among others, the condition

$$KNM = 0. \tag{47}$$

Assuming $N = M = 0$, which will be explained in the next section, the integrability condition reduces to

$$2\eta B + R_t = B_x + A_1 R + A_2\, \mathrm{D}R \tag{48}$$

$$K_t = -C_x - 2\eta C + A_2\, \mathrm{D}K - KA_1 - K\, \mathrm{D}A_2 \tag{49}$$

$$KB = A_{2x} - CR \tag{50}$$

$$K\, \mathrm{D}B - 2KA_2 R = A_{1x} - C\, \mathrm{D}R. \tag{51}$$

By the proper choice of the η dependence of the superfunctions A_1, A_2, B, C and their supersymmetric derivatives and by the proper choice of R and K we find the following.

(i) The fully supersymmetric KdV equation for which

$$R = q \qquad K = 1 \tag{52}$$

where B, A_1, A_2, C are defined by (16)-(19) respectively.

(ii) The fully supersymmetric MKdV equation for which

$$R = q \qquad K = -\mathrm{D}q \tag{53}$$

where B, C, A_1, A_2 are defined by (25)-(28) respectively.

4. The matrix supersymmetric extension of the AKNS representation

The system of the supersymmetric equations (44) and (45) is the supersymmetric projective representation of the AKNS. We now try to obtain its matrix version. Let us assume that formula (34) is valid also in this case. From the construction Γ is an odd superfield and therefore we should assume that ψ_1 is odd while ψ_2 is an even superfunction. Introducing such a factorised superfunction Γ to (44) and (45) with $N = M = 0$ we obtain

$$\psi_{1x}\psi_2 - \psi_1\psi_{2x} = 2\eta\psi_1\psi_2 + R\psi_2\psi_2 + K\psi_1\, \mathrm{D}\psi_1 \tag{54}$$

$$\psi_{1t}\psi_2 - \psi_1\psi_{2t} = B\psi_2\psi_2 + A_1\psi_1\psi_2 + A_2(\mathrm{D}\psi_1\psi_2 + \psi_1\, \mathrm{D}\psi_2) - C\psi_1\, \mathrm{D}\psi_1. \tag{55}$$

Notice that the assumption $N = M = 0$ is crucial. If we assume this is not true we cannot obtain the analogue of (54) and (55). We would like to split this system analogously as in the non-supersymmetric case. There are many ways to do it. We choose this way which reproduces the correct bosonic limit, e.g. gives us (35) and (36). We obtain

$$\psi_{1x} = \eta\psi_1 + R\psi_2 \qquad \psi_{1t} = B\psi_2 + A_2\, \mathrm{D}\psi_1 + h\psi_1 \tag{56}$$

$$-\psi_1\psi_{2x} = \eta\psi_1\psi_2 + K\psi_1\, \mathrm{D}\psi_1 \tag{57}$$

$$-\psi_1\psi_{2t} = A_1\psi_1\psi_2 + A_2\psi_1\, \mathrm{D}\psi_2 - C\psi_1\, \mathrm{D}\psi_1 - h\psi_1\psi_2, \tag{58}$$

where h is an arbitrary (on this level) even superfield. We cannot simply divide (57) and (58) by ψ_1 as in the non-supersymmetric case because ψ_1 is the odd superfunction. We can conclude only that

$$\psi_{2x} = -\eta\psi_2 - K\, \mathrm{D}\psi_1 + g_1\psi_1 \tag{59}$$

$$\psi_{2t} = -A_1\psi_2 + A_2\, \mathrm{D}\psi_2 + C\, \mathrm{D}\psi_1 + g_2\psi_1 + h\psi_2 \tag{60}$$

where g_1 and g_2 are an arbitrary (on this level) odd superfunctions. We fix these arbitrary superfunctions assuming the integrability of (59), (60) and (56). This condition give us (48)-(51) and additionally

$$Rg_2 = h_x + Bg_1 \tag{61}$$

$$g_{1t} - 2\eta g_2 - K\,\mathrm{D}h + 2\eta KA_2 = g_{2x} - A_1 g_1 + A_2\,\mathrm{D}g_1. \tag{62}$$

In that way (48)-(51) and (61) and (62) constitue the fully supersymmetric AKNS representation. We solved these quations for KdV and MKdV equations and hence we obtained the following matrix AKNS representation.

(i) Supersymmetric KdV equation:

$$g_1 = h = 0 \qquad g_2 = 2\eta R \tag{63}$$

$$\psi_x = \begin{pmatrix} \eta & R \\ 0 & -\eta \end{pmatrix}\psi + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\mathrm{D}\psi = S\psi + \tilde{S}\,\mathrm{D}\psi \tag{64}$$

$$\psi_t = \begin{pmatrix} 0 & B \\ 2\eta R & -A_1 \end{pmatrix}\psi + \begin{pmatrix} A_2 & 0 \\ C & A_2 \end{pmatrix}\mathrm{D}\psi = W\psi + \tilde{W}\,\mathrm{D}\psi. \tag{65}$$

(ii) Supersymmetric MKdV equation:

$$g_1 = -\eta q \qquad h = 2\eta q q_x \tag{66}$$

$$g_2 = \eta(q_{xx} - 2\eta q_x + 4\eta^2 q - 2\,\mathrm{D}q\,\mathrm{D}qq) \tag{67}$$

$$\psi_x = \begin{pmatrix} \eta & R \\ g_1 & -\eta \end{pmatrix}\psi + \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix}\psi \tag{68}$$

$$\psi_t = \begin{pmatrix} h & B \\ g_2 & h - A_1 \end{pmatrix}\psi + \begin{pmatrix} A_2 & 0 \\ C & A_2 \end{pmatrix}\psi. \tag{69}$$

5. The supersymmetric Bäcklund transformation

The Bäcklund transformation (BT) is an important tool in the theoretical understanding of a certain class of nonlinear equations of physical interest. They are very useful for the purpose of generating conservation laws and exact solutions of such equations, making use of known solutions as an input in the latter case. For this reason the construction of the BT is usually considered to be the ultimate goal. Various methods have been suggested for the derivation of this BT. For example, Konno and Wadati [20] showed how it is possible to derive it from the projective representation of AKNS while Levi *et al* [21] proved that BT can be interpreted as the gauge transformation of AKNS. Here we try to supersymmetrise these two methods for the KdV equation. First we derive this transformation from the supersymmetric projective AKNS representation. Next we show that the 'gauge-like' interpretation of this transformation is consistent with the invariance properties of the Riccati equation under this 'gauge-like' transformation.

Let us explain the transition from the Riccati equations to the BT and its 'gauge-like' interpretation in the non-supersymmetric case. This is achieved by the construction of a transformation Γ' satisying the same equation as (32) but with the new potential

$$\tilde{q}(x) = q(x) + f(\Gamma, \eta). \tag{70}$$

In the case of the KdV equation it is easy to show that $\tilde{q}$, defined by

$$\tilde{q} = q - 2\Gamma_x \tag{71}$$

where q is some solution of the KdV equation, is the new solution of the KdV equation if Γ is the solution of the Riccati equations (32) and (33). Now, using (71) to eliminate the Γ function in the Riccati equations we obtain the BT which for the KdV equation is

$$(\tilde{\omega} + \omega)_x = -2\eta^2 + \tfrac{1}{2}(\tilde{\omega} - \omega)^2 \tag{72}$$

$$(\omega + \tilde{\omega})_t = 2(\omega_x^2 + \omega_x\tilde{\omega}_x + \tilde{\omega}_x^2) - (\omega - \tilde{\omega})(\omega_{xx} - \tilde{\omega}_{xx}) \tag{73}$$

where $\omega_x = -q$, $\tilde{\omega}_x = -\tilde{q}$.

In order to obtain the transformed function Γ' the gauge transformation of the AKNS is very useful. Here the gauge transformation of AKNS means that, when ψ is transformed to ψ' by

$$\psi' = Z\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi \tag{74}$$

then ψ' satisfy the same AKNS matrix equations (37) and (38) with $\tilde{q}$ if

$$Z_x + ZS = S_1 Z \tag{75}$$

$$Z_t + ZW = W_1 Z \tag{76}$$

where S_1 and W_1 are the same matrices as in (37) and (38) in which q is replaced by $\tilde{q}$. The system of equations (75) and (76) is just the Bäcklund transformation in the matrix form and from the other side it corresponds to the gauge invariance of the connection form in the geometrical approach to the AKNS representation. From the gauge invariance of the AKNS representation follows the invariance of the Riccati equations. Indeed, under the gauge transformation (74) it is possible to find new Γ' defined by

$$\Gamma' = \frac{a\Gamma + b}{c\Gamma + d} \tag{77}$$

which satisfy the same Riccati equation (32) with $\tilde{q}$. Now we try to supersymmetrise this approach. We assume that in this case the formula (71) holds also. In order to check the validity of this assumption it is enough to introduce (71) to the supersymmetric KdV equation and observing that

$$\Gamma_t = \Gamma_{xxx} + 3\{q\, \mathrm{D}\Gamma_x + \mathrm{D}q\, \Gamma_x - 2\Gamma_x\, \mathrm{D}\Gamma_x\} \tag{78}$$

holds if Γ is the solution of the Riccati equation stemming from the supersymmetric KdV equation. After substituting our formula (71) into the Riccati equation this gives us the following BT:

$$(\omega + \tilde{\omega})_x = -\eta(\omega - \tilde{\omega}) + \xi\, \mathrm{D}(\omega - \tilde{\omega}) + 2\eta\xi + \tfrac{1}{2}(\omega - \tilde{\omega})\, \mathrm{D}(\omega - \tilde{\omega}) \tag{79}$$

$$\begin{aligned} -(\omega - \tilde{\omega})_t = {} & 2B + A_1[\tilde{\omega} - \omega - 2\xi] + A_2[\mathrm{D}(\tilde{\omega} - \omega) - \eta] \\ & - C[\tfrac{1}{2}(\omega - \tilde{\omega})\, \mathrm{D}(\omega - \tilde{\omega}) - \eta(\tilde{\omega} - \omega) - \xi\, \mathrm{D}(\tilde{\omega} - \omega) + 2\xi\eta] \end{aligned} \tag{80}$$

where $q = -\omega_x$, $\tilde{q} = -\tilde{\omega}_x$, $\xi = \varepsilon_0 + \theta\eta$, ε_0 is an arbitrary constant Grasmann number and B, A_1, A_2, C are defined by (16)-(19) respectively.

The same strategy can be applied to the supersymmetric MKdV equation and instead of (71) we should assume

$$\Gamma = \mathrm{D}^{-1} \tan \mathrm{D}^{-1} \tfrac{1}{2}(q - \tilde{q}) \tag{81}$$

where D^{-1} is the inverse operator to D satisfying

$$\mathrm{D}^{-1} = \mathrm{D} \int_{-\infty}^{x} = \int_{-\infty}^{x} \mathrm{D} \tag{82}$$

$$\mathrm{D}\mathrm{D}^{-1} = \mathrm{D}^{-1}\mathrm{D} = 1 \tag{83}$$

where q, $\tilde{q}$ are the old and new solutions of MKdV equation.

The method of deriving the BT does not look quite geometrical. Usually this BT as we mentioned earlier is interpreted as the gauge transformation of the AKNS representation. We show that in the supersymmetric case it is possible to construct such a gauge transformation which is consistent with our BT.

Let us explain how it is possible to define the gauge transformation of our AKNS representation in the supersymmetric case. We use the analogy to the non-supersymmetric case and assume that ψ transforms as

$$\psi' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi + \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \mathrm{D}\psi = E\psi + \tilde{E}\, \mathrm{D}\psi. \tag{84}$$

This reproduces the 'correct' bosonic limit. Assuming that ψ' satisfy the same supersymmetric matrix AKNS equations (64) and (65) with $\tilde{q}$ we get

$$E_x + ES + \tilde{E}\, \mathrm{D}S + \tilde{E}\bar{\tilde{S}}S = S_1 E + \tilde{S}_1\, \mathrm{D}E + \tilde{S}_1 \bar{\tilde{E}}S \tag{85}$$

$$\tilde{E}_x + E\tilde{S} + \tilde{E}\, \mathrm{D}\tilde{S} + \tilde{E}\bar{S} + \tilde{E}\bar{\tilde{S}}\bar{S} = S_1 \tilde{E} + \tilde{S}_1\, \mathrm{D}\tilde{E} + \tilde{S}_1 \bar{E} + \tilde{S}_1 \bar{\tilde{E}}\bar{S} \tag{86}$$

and similar formulae for the 't' part, where S_1, $\tilde{S}_1$ are the same matrices as in (64) but with $\tilde{q}$, the '—' over capitals is defined by (6).

We succeeded in obtaining the supersymmetric analogue of the gauge transformation of our AKNS representation or, in other words, the gauge transformation of the connection forms. The formulae (85) and (86) constitute also the 'x' part of the BT in matrix form. For technical reasons it is easier to work with the projective representation than with the gauge transformation if we wish to obtain BT. However these two approaches are equivalent in the sense that this gauge transformation create new

$$\Gamma' = \frac{a\psi_1 + b\psi_2 + \tilde{a}\, \mathrm{D}\psi_1 + \tilde{b}\, \mathrm{D}\psi_2}{c\psi_1 + d\psi_2 + \tilde{c}\, \mathrm{D}\psi_1 + \tilde{d}\, \mathrm{D}\psi_2} \tag{87}$$

which is the solution of the supersymmetric Riccati equation with $\tilde{q}$ by the use of the explicit form of (85) and (86). Let us mention that we just use this invariance for the construction of our Bäcklund transformation.

6. Conclusion

In this paper we show how it is possible to endow the fully supersymmetric soliton equations as KdV and MKdV in the supersymmetric AKNS representation. Let us mention that these equations themselves and their Lax representations are already known in literature but the method of their derivation is rather new. Hence the news value of the paper is rather limited. On the other hand, this approach can open the way to the construction of new supersymmetric equations as, for example, for the fully supersymmetric nonlinear Schrödinger equation.

Acknowledgments

The author would like to thank the anonymous referees for the constructive criticism which helped to improve the present paper.

This work was supported by CPBP 01.03.

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